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ON UNITAL $C(X)$-ALGEBRAS AND $C(X)$-VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

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Abstract. Let X be a compact Hausdorff space and let A be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital C^*-subalgebra of the centre of A. We consider the problem of characterizing the existence of a conditional expectation $E : A \to C(X)$ of finite index in terms of the underlying C^*-bundle of A over X. More precisely, we show that if A admits a $C(X)$-valued conditional expectation of finite index, then A is necessarily a continuous $C(X)$-algebra, and there exists a positive integer N such that every fibre A_x of A is finite-dimensional, with $\dim A_x \leq N$. We also give some sufficient conditions on A that ensure the existence of a $C(X)$-valued conditional expectation of finite index.

1. introduction

Let $B \subseteq A$ be two unital C^*-algebras with the same unit element. A *conditional expectation* (abbreviated by C.E.) from A to B is a completely positive contraction $E : A \to B$ such that $E(b) = b$ for all $b \in B$, and which is B-bilinear, i.e.

$$E(b_1 a b_2) = b_1 E(a) b_2$$

for all $a \in A$ and $b_1, b_2 \in B$. By a result of Y. Tomiyama (see [22, Theorem 1] or [4, Theorem II.6.10.2]), a map $E : A \to B$ is a C.E. if and only if E is a projection of norm one.

If $E(a^*a) = 0$ $(a \in A)$ implies $a = 0$, E is said to be *faithful*. Every faithful conditional expectation $E : A \to B$ introduces a pre-Hilbert B-module structure on A, whose inner product is defined by

$$\langle a_1, a_2 \rangle_E := E(a_1^* a_2) \qquad (a_1, a_2 \in A). \tag{1.1}$$

The notion of finite index was introduced by V. F. R. Jones [14] in order to classify the subfactors of a type II_1 factor. Soon afterwards H. Kosaki [16] extended the Jones index theory to arbitrary factors. In order to generalize the results of [14, 16], M. Pimsner and S. Popa introduced in [19, 20] a definition for conditional expectations of finite index in the context of W^*-algebras: There must exist a constant $K \geq 1$ such that the map $K \cdot E - \mathrm{id}_A$ is positive on A. Then, following the idea of M. Baillet, Y. Denizeau and J.-F. Havet (see [3]), the index of E can be defined in the following way: Since the map $K \cdot E - \mathrm{id}_A$ is positive, E defines a (complete) Hilbert B-module structure on A, with respect to the inner product

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(1.1). If $\{x_i\}$ is a quasi-orthonormal basis in A, the *index* of E is the sum $\sum_{i=1}^{\infty} x_i^* x_i$, with respect to the ultraweak topology.

Y. Watatani also considered C.E. of (algebraically) finite index, when the original C^*-algebra A is a finitely generated Hilbert C^*-module over B (see [23]).

The results of M. Baillet, Y. Denizeau and J.-F. Havet in [3] also indicated that there might occur some difficulties in order to extend the notion of "finite index" for conditional expectations of C^*-algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [11]. The main result of their paper is [11, Theorem 1]:

Theorem 1.1 (M. Frank and E. Kirchberg). *For a C.E. $E : A \to B$, where $B \subseteq A$ are unital C^*-algebras with the same unit element, the following conditions are equivalent:*

(i) *There exists a constant $K \geq 1$ such that the map $K \cdot E - \mathrm{id}_A$ is positive.*
(ii) *There exists a constant $L \geq 1$ such that the map $L \cdot E - \mathrm{id}_A$ is completely positive.*
(iii) *A becomes a (complete) Hilbert B-module when equipped with the inner product (1.1).*

Moreover, if

$$K(E) := \inf\{K \geq 1 \ : \ K \cdot E - \mathrm{id}_A \text{ is positive}\},$$
$$L(E) := \inf\{L \geq 1 \ : \ L \cdot E - \mathrm{id}_A \text{ is completely positive}\},$$

with $K(E) = \infty$ or $L(E) = \infty$ if no such number K or L exists, then

$$K(E) \leq L(E) \leq \lfloor K(E) \rfloor K(E),$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on C^*-algebras to be of finite index:

Definition 1.2. If $B \subseteq A$ are two unital C^*-algebras with the same unit element, then a C.E. $E : A \to B$ is said to be of *finite index* (abbreviated C.E.F.I.) if E satisfies one of the equivalent conditions of Theorem 1.1.

In this case the index value of E can be calculated in the enveloping von Neumann algebra A^{**} (see [11, Definition 3.1]).

For a unital inclusion $A \subseteq B$ of unital C^*-algebras we introduce the following constant

$$K(A, B) := \inf\{K(E) \ : \ E : A \to B \text{ is C.E.F.I.}\},$$

with $K(A, B) = \infty$, if no such C.E.F.I. exists. This constant will play an important role in this paper.

More recently, A. Pavlov and E. Troitsky considered in [17] the problem of existence of a C.E.F.I. $E : C(Y) \to C(X)$ for a unital inclusion $\varphi : C(X) \hookrightarrow C(Y)$ of unital commutative C^*-algebras. The main result of their paper is [17, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map $\varphi_* : Y \to X$ is a *branched covering*. This means that φ_* is an open map with uniformly bounded number of pre-images (i.e. $\sup_{x \in X} |\varphi_*^{-1}(x)| < \infty$). This result motivated A. Pavlov and E. Troitsky to define the *noncommutative branched coverings*, as unital inclusion $B \subseteq A$ of unital C^*-algebras such that there exists a C.E.F.I. from A to B (see [17, Definition 1.2]).

Using the above inclusion $\varphi : C(X) \hookrightarrow C(Y)$ we may consider $C(Y)$ as a $C(X)$-algebra. Then the map φ_* is open if and only if $C(Y)$ is a continuous $C(X)$-algebra, and φ_* has uniformly bounded number of pre-images if and only if $C(Y)$ is subhomogeneous $C(X)$-algebra. This means that there exists a positive integer N such that every fibre $C(Y)_x$ of $C(Y)$ is finite-dimensional with $\dim C(Y)_x \leq N$ (see Section 2). Therefore, we can restate [17, Theorem 1.1] in terms of $C(X)$-algebras as follows:

Theorem 1.3 (A. Pavlov and E. Troitsky)**.** *Let A be a unital commutative $C(X)$-algebra, where $C(X)$ is embedded as a unital C^*-subalgebra of A. Then A admits a $C(X)$-valued C.E.F.I. if and only if A is a continuous subhomogeneous $C(X)$-algebra.*

The purpose of the present paper is to consider a possible extension of Theorem 1.3 to the case when A is an arbitrary (not necessarily commutative) unital $C(X)$-algebra. The necessary condition for the existence of a $C(X)$-valued C.E.F.I. appears to be identical to the one of Theorem 1.3:

Theorem 1.4. *Let A be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital C^*-subalgebra of the centre of A. If A admits a $C(X)$-valued C.E.F.I., then A is a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case the following inequality holds:*

$$K(A, C(X)) \geq r(A),$$

where $r(A)$ is the rank of A, i.e.

$$r(A) = \max \left\{ \sum_{[\pi_x] \in \widehat{A_x}} \dim \pi_x \;:\; x \in X \right\}.$$

We shall prove Theorem 1.4 in Section 3. At the moment we do not know if the converse of Theorem 1.4 also holds. However, if all the fibres of a continuous unital $C(X)$-algebra A are $*$-isomorphic to the same finite-dimensional C^*-algebra (i.e. A is a homogeneous $C(X)$-algebra), then there exists a unique C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \mathrm{id}_A$ is positive (Proposition 3.4). In particular, we have the equality $K(A, C(X)) = r(A)$ in this case. Also, a direct consequence of this fact is that any unital $C(X)$-algebra A which can be embedded as a $C(X)$-subalgebra of some continuous homogeneous unital $C(X)$-algebra also admits a $C(X)$-valued C.E.F.I.. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital $C(X)$-algebra A over a second-countable compact Hausdorff space X with fibres $M_2(\mathbb{C})$ or $\mathbb{C}$ which admits a $C(X)$-valued C.E.F.I., but which cannot be embedded as a $C(X)$-subalgebra into any continuous homogeneous unital $C(X)$-algebra (Example 3.6). At the end of this paper we also show that any continuous unital $C(X)$-algebra A of rank 2 admits a C.E. $E : A \to C(X)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive (Proposition 3.7). In particular, the equality $K(A, C(X)) = r(A)$ also holds in this class of $C(X)$-algebras.

2. Notation and preliminaries

Throughout this paper A will be a C^*-algebra. We denote by A_{sa} and A_+ the self-adjoint and the positive parts of A. The centre of A is denoted by $Z(A)$. By

$\hat{A}$ and $\mathrm{Prim}(A)$ we respectively denote the *spectrum* of A (i.e. the set of all classes of irreducible representations of A) and the *primitive spectrum* of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology. By a *dimension* of $[\pi] \in \hat{A}$, which is denoted by $\dim \pi$, we mean the dimension of the underlying Hilbert space of some representative of $[\pi]$.

Let X be a compact Hausdorff space. For each point $x \in X$ let

$$C_x(X) := \{f \in C(X) \ : \ f(x) = 0\}$$

be the corresponding maximal ideal of $C(X)$.

Definition 2.1. A $C(X)$-*algebra* is a C^*-algebra A endowed with a unital $*$-homomorphism ψ_A from $C(X)$ to the centre of the multiplier algebra of A.

Remark 2.2. Given $f \in C(X)$ and $a \in A$, we write fa for the product $\psi_A(f) \cdot a$ if no confusion is possible.

There is a natural connection between $C(X)$-algebras and upper semicontinuous C^*-bundles over X. We first give a formal definition of such bundles:

Definition 2.3. Following [24] by an *upper semicontinuous C^*-bundle* we mean a triple $\mathfrak{A} = (p, \mathcal{A}, X)$ where $\mathcal{A}$ is a topological space with a continuous open surjection $p : \mathcal{A} \to X$, together with operations and norms making each *fibre* $\mathcal{A}_x := p^{-1}(x)$ into a C^*-algebra, such that the following conditions are satisfied:

- (A1) The maps $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}$ given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous ($\mathcal{A} \times_X \mathcal{A}$ denotes the Whitney sum over X).
- (A2) The map $\mathcal{A} \to \mathbb{R}$, defined by norm on each fibre, is upper semicontinuous.
- (A3) If $x \in X$ and if (a_α) is a net in $\mathcal{A}$ such that $\|a_\alpha\| \to 0$ and $p(a_\alpha) \to x$ in X, then $a_\alpha \to 0_x$ in $\mathcal{A}$ (0_x denotes the zero-element of $\mathcal{A}_x$).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that $\mathfrak{A}$ is a *continuous* C^*-bundle.

By a *section* of an upper semicontinuous C^*-bundle $\mathfrak{A}$ we mean a map $s : X \to \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. We denote by $\Gamma(\mathfrak{A})$ the set of all continuous sections of $\mathfrak{A}$. Then $\Gamma(\mathfrak{A})$ becomes a $C(X)$-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a $C(X)$-algebra A, one can always associate an upper semicontinuous C^*-bundle $\mathfrak{A}$ over X such that $A \cong \Gamma(\mathfrak{A})$, as follows. Set $J_x := C_x(X) \cdot A$ and note that J_x is a closed two-sided ideal in A (by Cohen factorization theorem [7], [6, Theorem A.6.2])). The quotient $A_x := A/J_x$ is called the *fibre* at the point x, and we denote by a_x the image in A_x of an element $a \in A$. Let

$$\mathcal{A} := \bigsqcup_{x \in X} A_x,$$

and let $p : \mathcal{A} \to X$ be the canonical associated projection. For $a \in A$ we define the map $\hat{a} : X \to \mathcal{A}$ by $\hat{a}(x) := a_x$, and let $\Omega := \{\hat{a} \ : \ a \in A\}$. Since for each $a \in A$ we have

$$\|a_x\| = \inf\{\| [1 - f + f(x)] \cdot a\| \ : \ f \in C(X)\},$$

the norm function $x \mapsto \|a_x\|$ is upper semicontinuous on X. Hence, by Fell's theorem [24, Theorem C.25] there exists a unique topology on $\mathcal{A}$ for which $\mathfrak{A} := (p, \mathcal{A}, X)$

becomes an upper semicontinuous C^*-bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee's theorem [24, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the *generalized Gelfand transform* $\mathcal{G} : a \in A \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$-algebras, from A onto $\Gamma(\mathfrak{A})$.

Definition 2.4. Let A be a $C(X)$-algebra. If all the norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are continuous on X, we say that A is a *continuous $C(X)$-algebra.*

Note that the $C(X)$-algebra A is continuous if and only if $\mathfrak{A}$ is continuous as a C^*-bundle.

The C^*-algebra A is said to be

- (n-)homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \widehat{A}$,
- (n-)subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi]\in\widehat{A}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$-algebras. To do this, first recall that if D is a finite-dimensional C^*-algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \dots, p_m \in Z(D)$ with $\sum_{i=1}^m p_i = 1_D$, such that

$$D = p_1 D \oplus \cdots \oplus p_m D, \tag{2.1}$$

and each $p_i D$ is $*$-isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [21, Theorem I.11.9]). We define the *rank* of D as

$$r(D) := \sum_{i=1}^m n_i = \sum_{[\pi]\in\hat{D}} \dim \pi.$$

Definition 2.5. Let A be a $C(X)$-algebra. We say that A is

- *homogeneous* all the fibres of A are $*$-isomorphic to the same finite-dimensional C^*-algebra.
- *subhomogeneous* if there exists a positive integer N such that every fibre A_x of A is finite-dimensional with $\dim A_x \le N$.

Remark 2.6. Let A be a $C(X)$-algebra.

(i) A is subhomogeneous if and only if

$$r(A) := \sup\{r(A_x) \ : x \in X\} < \infty$$

As in the finite-dimensional case, we call the number $r(A)$ the *rank* of A.

(ii) If A is continuous and homogeneous, then by [10, Lemma 3.1] the underlying C^*-bundle $\mathfrak{A}$ is locally trivial.

3. Results

Remark 3.1. If A is a unital $C(X)$-algebra, we always assume in this section that the map $\psi_A : C(X) \to Z(A)$ is injective, so that we can identify $C(X)$ with the unital C^*-subalgebra $\psi_A(C(X))$ of $Z(A)$.

In order to prove Theorem 1.4 we shall need the following two auxiliary results.

Lemma 3.2. *Let D be a unital C^*-algebra. Then $K(D, \mathbb{C}) := K(D, \mathbb{C}1_D) < \infty$ if and only if D is finite-dimensional. In this case we have:*

(i) *The constant $K(\omega)$ is finite for every faithful state ω on D, which we identify with the corresponding faithful C.E.*

$$d \in D \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$

(ii) $K(D,\mathbb{C}) = r(D)$. *Moreover, there exists a unique state* τ *on* D *such that*

$$r(D) \cdot \tau(d) 1_D \geq d \quad \text{for all } d \in D_+. \tag{3.1}$$

Proof. The equivalence $K(D,\mathbb{C}) < \infty \Leftrightarrow \dim D < \infty$ follows from [13, Lemma 4.5]. Hence, suppose that D is finite-dimensional and let ω be a faithful state on D. The proof will now proceed in two steps.

Step 1. Assume that D is simple, i.e. $D = M_n(\mathbb{C})$ for some n. If $\mathrm{tr}(\cdot)$ is the standard trace of $M_n(\mathbb{C})$, then there exists a strictly positive matrix $a \in M_n(\mathbb{C})$ with $\mathrm{tr}(a) = 1$ such that

$$\omega(d) = \mathrm{tr}(ad) \quad (d \in M_n(\mathbb{C})).$$

Let $a = u^* \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \cdot u$ be a diagonalisation of a, where $u \in M_n(\mathbb{C})$ is a unitary and $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of a. Then for all $d \in M_n(\mathbb{C})$ one has

$$\omega(u^* du) = \mathrm{tr}(au^* du) = \mathrm{tr}(uau^* d) = \mathrm{tr}(\mathrm{diag}(\lambda_1, \ldots, \lambda_n) d). \tag{3.2}$$

The constant $K(\omega)$ is by definition the smallest $K \geq 1$ satisfying

$$K \cdot \omega(d) 1_D \geq d \quad \text{for all } d \in D_+. \tag{3.3}$$

Thus, (3.2) and (3.3) for rank 1 projections in D imply that

$$K(\omega) = \max\{\lambda_i^{-1} \ : \ 1 \leq i \leq n\}.$$

As $1 = \omega(1) = \sum_{i=1}^n \lambda_i$, one has $K(\omega) \geq n$ for any faithful state ω on D. Also, $K(\omega) = n$ if and only if $\omega = \tau := \frac{1}{n}\mathrm{tr}(\cdot)$. In particular, if $D = M_n(\mathbb{C})$, we have $K(D,\mathbb{C}) = r(D) = n$, and τ is the unique state on D satisfying (3.1).

Step 2. Suppose that D is an arbitrary finite-dimensional C^*-algebra. We decompose D as in (2.1). For each $1 \leq i \leq m$

$$\omega_i(p_i d) := \frac{1}{\omega(p_i)} \cdot \omega(p_i d) \quad (d \in D)$$

defines a faithful state on $p_i D$. By Step 1 we have $n_i \leq K(\omega_i) < \infty$ for all $1 \leq i \leq m$. Put

$$K_\omega := \max\left\{\frac{K(\omega_i)}{\omega(p_i)} \ : \ 1 \leq i \leq m\right\}.$$

We claim that $K(\omega) = K_\omega$. Indeed, for all $d \in D_+$ we have

$$\begin{aligned} K_\omega \cdot \omega(d) 1_D &= \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_i d) 1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_i d) p_i \\ &\geq \sum_{i=1}^m p_i d = d, \end{aligned}$$

which shows $K(\omega) \leq K_\omega$. On the other hand, for each $d \in D_+$ we have

$$[\omega(p_i) K(\omega)] \cdot \omega_i(p_i d) p_i \geq p_i d\,,$$

so that

$$\omega(p_i) K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m). \tag{3.4}$$

This shows $K(\omega) = K_\omega$, as wanted. Also,

$$K(\omega) = \sum_{i=1}^{m} \omega(p_i) K(\omega) \geq \sum_{i=1}^{m} K(\omega_i) \geq \sum_{i=1}^{m} n_i = r(D)\,,$$

so that $K(D, \mathbb{C}) \geq r(D)$.

It remains to show that there exists a unique state τ on D satisfying (3.1). To do this, suppose that $r(D) = n$, and for each $1 \leq i \leq m$ let τ_i be the only faithful tracial state on $p_i D \cong M_{n_i}(\mathbb{C})$. Define the state τ on D by

$$\tau(d) := \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_i d) \quad (d \in D). \tag{3.5}$$

As $\tau(p_i) = \frac{n_i}{n}$ and $K(\tau_i) = n_i$ for all $1 \leq i \leq m$, we have $K(\tau) = K_\tau = n$. In particular, $K(D, \mathbb{C}) = n = r(D)$.

To show the uniqueness of this state τ, suppose that ω is another state on D with $K(\omega) = n$. Then using (3.4) we have

$$\sum_{i=1}^{m} K(\omega_i) \leq \sum_{i=1}^{m} \omega(p_i) K(\omega) = K(\omega) = n.$$

But since $K(\omega_i) \geq n_i$ and $\sum_{i=1}^{m} n_i = n$, we must have $K(\omega_i) = n_i$ for all $1 \leq i \leq m$. By the uniqueness part of Step 1 we conclude that

$$\omega_i = \tau_i \quad \text{for all} \;\; 1 \leq i \leq m. \tag{3.6}$$

Also, $K_\omega = K(\omega) = n$ and $K(\omega_i) = n_i$ imply $\omega(p_i) \geq \frac{n_i}{n}$ for all $1 \leq i \leq m$. Since ω is a state on D and $\sum_{i=1}^{m} p_i = 1_D$, we must have

$$\omega(p_i) = \frac{n_i}{n} \quad \text{for all} \;\; 1 \leq i \leq m. \tag{3.7}$$

Finally, (3.6) and (3.7) imply that

$$\omega(d) = \sum_{i=1}^{m} \omega(p_i) \omega_i(p_i d) = \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_i d) = \tau(d),$$

for all $d \in D$, which finishes the proof. $\square$

Proposition 3.3. *Let A be a unital $C(X)$-algebra. If A admits a faithful $C(X)$-valued C.E., then A is a continuous $C(X)$-algebra.*

Proof. This can be deduced from [5, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are lower semicontinuous on X. To prove this, let $E : A \to C(X)$ be a faithful C.E. and let $L^2(A, E)$ be the completion of the pre-Hilbert $C(X)$-module A, with respect to the inner product (1.1). For $a \in A$ let $\Phi(a) : L^2(A, E) \to L^2(A, E)$ denote the continuous extension of the left multiplication map $a_1 \mapsto a a_1$ $(a \in A)$. Since E is faithful and since

$$\begin{aligned} \langle \Phi(a)(a_1), a_2 \rangle_E &= \langle a a_1, a_2 \rangle_E = E(a_1^* a^* a_2) = \langle a_1, a^* a_2 \rangle_E \\ &= \langle a_1, \Phi(a^*)(a_2) \rangle_E, \end{aligned}$$

for all $a_1, a_2 \in A$, the map Φ defines an injective $C(X)$-linear morphism from A to the $C(X)$-algebra $\mathbb{B}_{C(X)}(L^2(A,E))$ of bounded adjointable $C(X)$-linear operators on $L^2(A,E)$. Therefore, for $a \in A$ and $x \in X$ we have

$$\begin{aligned} \|a_x\| &= \|\Phi(a)_x\| \\ &= \sup\{|\langle \Phi(a)(a_1), a_2\rangle_E(x)| \ : \ a_1, a_2 \in A, \ \|a_1\|_E = \|a_2\|_E = 1\} \\ &= \sup\{|E(a_1^* a^* a_2)(x)| \ : \ a_1, a_2 \in A, \ \|a_1\|_E = \|a_2\|_E = 1 \ \}. \end{aligned}$$

In particular, the function $x \mapsto \|a_x\|$ is a supremum of continuous functions $x \mapsto |E(a_1^* a^* a_2)(x)|$ ($\|a_1\|_E = \|a_2\|_E = 1$), so it must be lower semicontinuous on X. □

Proof of Theorem 1.4. Let $E : A \to C(X)$ be a C.E.F.I.. As the conditional expectation E is faithful, Proposition 3.3 implies that the $C(X)$-algebra A is continuous (note that in this case $(A, \langle\cdot,\cdot\rangle_E)$ is already a complete Hilbert $C(X)$-module by Theorem 1.1). It remains to show that each fibre A_x ($x \in X$) is finite-dimensional and satisfies $r(A_x) \le K(E)$. Indeed, for a fixed point $x \in X$ and $\varepsilon > 0$,

$$\omega_x : a_x \mapsto E(a)(x)$$

defines a state on a fibre A_x satisfying

$$(K(E) + \varepsilon) \cdot \omega_x(a_x) 1_x \ge a_x$$

for all $a_x \in (A_x)_+$. Lemma 3.2 now yields $r(A_x) \le K(E)$, as wanted. □

We shall now give some sufficient conditions on a continuous unital subhomogeneous $C(X)$-algebra A to ensure the existence of a $C(X)$-valued C.E.F.I..

Proposition 3.4. *Every continuous homogeneous unital $C(X)$-algebra A admits a unique C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \mathrm{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.*

Proof. The construction of such a C.E. $E : A \to C(X)$ can be deduced from the proof of [13, Lemma 4.6]. But we include here the main steps of the proof for completeness. By assumption all fibres of A are $*$-isomorphic to a fixed finite-dimensional C^*-algebra D. Suppose that $r(D) = n$, and let τ be a state on D defined by (3.5). It is easy check that τ is invariant under the group $\mathrm{Aut}(D)$ of $*$-automorphisms of D. Since the $C(X)$-algebra A is continuous and homogeneous, its underlying bundle $\mathfrak{A}$ is locally trivial by Remark 2.6. Hence, there exists an open covering $\{U_\alpha\}$ of X such that $\Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong U_\alpha \times D$, where

- Φ_α is an isomorphism of C^*-bundles, and
- $\mathfrak{A}|_U$ is the restriction bundle over a subset $U \subseteq X$.

Fix an element $a \in A$. For $x \in X$ choose an index α such that $x \in U_\alpha$, and define

$$E(a)(x) := \tau(\Phi_\alpha(a_x)).$$

Since τ is invariant under the group $\mathrm{Aut}(D)$, the value $E(a)(x)$ is well defined, and the local triviality of $\mathfrak{A}$ implies that the function $E(a) : x \mapsto E(a)(x)$ is continuous on X. It is now easy to see that the map $E : a \to E(a)$ defines a $C(X)$-valued C.E.F.I. on A. Moreover, by (3.1) we have

$$n \cdot E(a)(x) 1_x \ge a_x, \quad \text{for all } a \in A_+ \text{ and } x \in X.$$

Thus, the map $n \cdot E - \mathrm{id}_A$ is positive and E is the only C.E. with this property (Lemma 3.2). In particular, $K(A, C(X)) \le r(A)$, so Theorem 1.4 yields that $K(A, C(X)) = n$. □

Corollary 3.5. *If the unital $C(X)$-algebra A admits a $C(X)$-linear embedding into some homogeneous continuous unital $C(X)$-algebra A', then A admits a $C(X)$-valued C.E.F.I..*

Proof. By Proposition 3.4 there exists a C.E. $E' : A' \to C(X)$ of finite index. Then the restriction $E'|_A : A \to C(X)$ defines a convenient C.E.F.I.. □

Note that the embedding condition of Corollary 3.5 is not necessary for the existence of a $C(X)$-valued C.E.F.I.. Indeed, in Example 3.6 we show that there exists a continuous unital $C(X)$-algebra A of rank 2 which does not admit a $C(X)$-linear embedding into any continuous homogeneous unital $C(X)$-algebra. On the other hand, a direct consequence of Proposition 3.7 is that A admits a $C(X)$-valued C.E.F.I..

To do this, first recall that a C^*-algebra A is said to be *central* if it satisfies the following two conditions:

(i) A is *quasi-central* (i.e. no primitive ideal of A contains $Z(A)$);
(ii) If $P, Q \in \mathrm{Prim}(A)$ and $P \cap Z(A) = Q \cap Z(A)$, then $P = Q$

(see [1, 8, 12, 15]). By [8, Proposition 3] a quasi-central C^*-algebra A is central if and only if $\mathrm{Prim}(A)$ is Hausdorff.

Example 3.6. By [18, Example 3.5] there exists a continuous $M_2(\mathbb{C})$-bundle $\mathfrak{A}_0$ over the second countable locally compact space $X_0 := \bigsqcup_{n=1}^{\infty} \mathbb{C}P^n$, where $\mathbb{C}P^n$ is the complex projective space of dimension n, which is not of finite type (that is, X_0 does not admit a finite open cover $\{U_i\}$ such that each restriction bundle $\mathfrak{A}_0|_{U_i}$ is trivial, as a C^*-bundle). Let A_0 be the C^*-algebra $\Gamma_0(\mathfrak{A}_0)$ consisting of all continuous sections of $\mathfrak{A}_0$ which vanish at infinity. Then A_0 is a 2-homogeneous C^*-algebra with $\mathrm{Prim}(A_0) = X_0$. In particular A_0 is a central C^*-algebra with centre $C_0(X_0)$. Let $X := X_0 \sqcup \{\infty\}$ be the one-point compactification of X_0, and let A be the minimal unitisation of A_0. By [8, Proposition 3] (or [12, Proposition 3.12]) A is also a central C^*-algebra with $\mathrm{Prim}(A) = X$ and centre $C(X)$. In particular, by [4, II.6.5.8] all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are continuous on X, so that A is a continuous unital $C(X)$-algebra with fibres $A_x = M_2(\mathbb{C})$ $(x \in X_0)$ and $A_\infty = \mathbb{C}$. Suppose that A is $C(X)$-subalgebra of some continuous homogeneous $C(X)$-algebra A'. Then the underlying C^*-bundle $\mathfrak{A}$ of A over X is a C^*-subbundle of the underlying C^*-bundle $\mathfrak{A}'$ of A' over X. Since A' is continuous and homogeneous, $\mathfrak{A}'$ is locally trivial by Remark 2.6. Hence, since X is compact, $\mathfrak{A}'$ is of finite type. Using [18, Lemma 2.6] we conclude that $\mathfrak{A}$ is of finite type as a vector bundle. In particular, $\mathfrak{A}_0$ is of finite type as a vector bundle, since $\mathfrak{A}_0 = \mathfrak{A}|_{X_0}$. As $\mathfrak{A}_0$ is a $M_2(\mathbb{C})$-bundle, this implies by [18, Proposition 2.9] that $\mathfrak{A}_0$ is also of finite type as a C^*-bundle; a contradiction.

On the other hand, the $C(X)$-algebra A of Example 3.6 also admits a $C(X)$-valued C.E.F.I.. This follows from the following more general fact:

Proposition 3.7. *Let A be a continuous unital $C(X)$-algebra. If $r(A) = 2$, then there exists a conditional expectation $E : A \to C(X)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.*

In order to prove Proposition 3.7, let us first make the following observation:

Lemma 3.8. *Let A be a unital $C(X)$-algebra and let $a \in A_{sa}$. For each point $x \in X$ let $\lambda_{\max}(a)$ and $\lambda_{\min}(a)$ respectively denote the largest and the smallest numbers in* *the spectrum of a_x. Then the functions $x \mapsto \lambda_{\max}(a_x)$ and $x \mapsto \lambda_{\min}(a_x)$ are upper semicontinuous on X. Furthermore, these functions are continuous on X, whenever A is a continuous $C(X)$-algebra.*

Proof. This follows directly from the equations

$$\lambda_{\max}(a_x) = \| \|a\| 1_x + a_x\| - \|a\| \quad \text{and} \quad \lambda_{\min}(a_x) = \|a\| - \| \|a\| 1_x - a_x\|.$$

□

Proof of Proposition 3.7. As $r(A) = 2$, any fibre A_x is isomorphic to $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$. Therefore, for each point $x \in X$ we can choose a unital embedding $\varphi_x : A_x \hookrightarrow M_2(\mathbb{C})$. For $a \in A$ and $x \in X$ we define

$$E(a)(x) := \frac{1}{2} \mathrm{tr}(\varphi_x(a_x)).$$

Obviously $E(a)$ is a $C(X)$-linear map. If $a \in A_{sa}$, note that

$$E(a)(x) = \frac{1}{2}(\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \tag{3.8}$$

for all $x \in X$. By Remark 3.8, $E(a)$ is a continuous function on X for all $a \in A_{sa}$. As A is the linear span of A_{sa}, we conclude that $E(a) \in C(X)$ for all $a \in A$. Therefore, E defines a C. E. from A onto $C(X)$. Further, by (3.8) for all $a \in A_+$ and $x \in X$ we have

$$2 \cdot E(a)(x) 1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.$$

This shows that the map $2 \cdot E - \mathrm{id}_A$ is positive, so that $K(A, C(X)) = 2$ by Theorem 1.4. □

Let A be a unital C^*-algebra and let $\check{Z}$ be the maximal ideal space of $Z(A)$. We may consider A as a $C(\check{Z})$-algebra, with respect to the action

$$f \cdot a := \mathcal{G}^{-1}(f) a \quad (f \in C(X),\ a \in A),$$

where $\mathcal{G} : Z(A) \to C(\check{Z})$ is the Gelfand transform. We say that A is *quasi-standard* if A is a continuous $C(\check{Z})$-algebra and each (Glimm) ideal $J_x = C_x(\check{Z})A$ is primal (see [2]).

Corollary 3.9. *For a unital C^*-algebra A the following conditions are equivalent:*

(i) *There exist a C.E. $E : A \to Z(A)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive.*
(ii) *A is either commutative or quasi-standard and 2-subhomogeneous.*

Proof. (i) $\Rightarrow$ (ii). Suppose that there exists a C.E. $E : A \to Z(A)$ such that the map $2 \cdot E - \mathrm{id}_A$ is positive. Then by Theorem 1.4 A is a continuous $C(\check{Z})$-algebra and $r(A_x) \leq 2$ for all $x \in \check{Z}$. In particular, A as a C^*-algebra is n-subhomogeneous, where $n \in \{1, 2\}$. Hence, by [13, Proposition 4.1] every Glimm ideal of A is primal. Also, $n = 1$ if and only if A is commutative.

(ii) $\Rightarrow$ (i). If A is commutative we have nothing to prove, so suppose that A is quasi-standard and 2-subhomogeneous. Then by [9, Corollary 1, p. 388] for each point $x \in X$ we have

$$r(A_x) = \sum_{[\pi_x] \in \widehat{A_x}} \dim \pi_x \leq 2.$$

It remains to apply Proposition 3.7. □

Remark 3.10. At the end of this paper we note that every separable continuous unital $C(X)$-algebra A admits a faithful C.E. $E : A \to C(X)$ (see e.g. [5]). In particular, this result applies to continuous subhomogeneous unital $C(X)$-algebras, when X is second-countable. In this case for each point $x \in X$, the map $E_x : a_x \mapsto E(a)(x)$ defines a faithful state on A_x, so Lemma 3.2 implies $K(E_x) < \infty$. However, this does not imply that E is of finite index. That is, it may happen that $\sup_{x\in X} K(E_x) = \infty$. Consider for instance the following example:

- Let X be the closed compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $[0, 1]$.
- Let A be the continuous $C(X)$-subalgebra of $C(X) \oplus C(X)$ consisting of all pairs $(f, g) \in C(X) \oplus C(X)$ such that $f(0) = g(0)$.
- Let $E : A \to C(X)$ be a C.E. fixed by the relations

$$E(f \oplus g)\left(\frac{1}{n}\right) = \begin{cases} \frac{n}{n+1} f(\frac{1}{n}) + \frac{1}{n+1} g(\frac{1}{n}) & \text{if } n \text{ is odd} \\ \\ \frac{1}{n+1} f(\frac{1}{n}) + \frac{n}{n+1} g(\frac{1}{n}) & \text{otherwise} \end{cases}$$

  where $(f, g) \in A$.

Then E is a faithful C.E. which is not of finite index. Indeed, one has

$$E(f \oplus 0)\left(\frac{1}{2n}\right) = \frac{1}{2n+1} f\left(\frac{1}{2n}\right)$$

for all $f \in C_0(X \setminus \{0\})$ and all integers $n \in \mathbb{N}$. Consequently, a convenient constant K would satisfy $K \geq 2n + 1$ for all $n \in \mathbb{N}$, which is impossible.

We end this paper with some unresolved problems:

Problem 3.11. Is the converse of Theorem 1.4 also true? Moreover, does every continuous subhomogeneous unital $C(X)$-algebra A admit a C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \mathrm{id}_A$ is positive? In particular, do we always have $K(A, C(X)) = r(A)$?

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